Small Disturbance Phenomena in Rarefied Gas Fields

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Nomenclature

a	=	particle radius for mass m
c	=	$(RT_0)^{1/2}$
c_l	=	$(5/3RT_0)^{1/2}$
E	=	energy
f_0	=	$\{\rho_0/(2\pi RT_0)^{3/2}\}\exp\{-(\xi^2+\eta^2+\zeta^2/2RT_0)\}$
f	=	$f_0(1+\Sigma\epsilon_i)$
F	=	force
k	=	const, also, k', k''
L_0	=	propagation distance
m	=	mass
n_0	=	no. of field particle collisions
n_i	=	$n_0/n_0'(n_0' < n_0)$
T	=	temperature
u, v	=	disturbance velocities
α	=	v_0/c
ϵ_1	=	$(\xi^2+\eta^2+\zeta^2/2RT_0)\theta_w$
ϵ_2	=	$-2\theta_x$
ϵ_3	=	$\xi u/RT_0$
ξ, η, ζ	=	molecular velocities
ρ	=	density
θ	=	disturbance temperature
σ_{xy}	=	shear stress
σ_x	=	dimensionless normal stress
p	=	disturbance pressure, $\Delta p/p_0$
P, Q	=	characteristic quantities
$P_{xy}, P_{\phi r}$	=	shear stress
P_{yy}	=	normal stress
q	=	heat flux, q/p_0c
q^2	=	thermal disturbance force constant
r	=	radial distance
t	=	time, dimensionless time, tc/L_0
τ_0	=	relaxation time, μ_0/p_0
μ	=	viscosity
ω	=	frequency

Subscripts

1	=	central body
l	=	longitudinal
0	=	equilibrium state
t	=	transverse
w	=	wall condition

Introduction

RECENTLY, Chester[1] has discussed the possibility of the existence of thermal wave propagation phenomena in dielectric solids similar to the phenomena of second sound observed in liquid helium. A similar phenomenon, also controlled by an inverse relaxation time relationship, is suggested by recent analyses of rarefied gas field phenomena.[2–5]

The small disturbance propagation equations for gases (acoustic equations) based on the Navier-Stokes equations describe only the phenomena associated with small velocity and density (or pressure) disturbances, phenomena for which the absorption coefficient varies directly with the relaxation time. The thermal disturbances in gases described by these equations are spread by diffusion.[6]

However, current investigations of rarefied gas phenomena based on approximate solutions of the Maxwell-Boltzmann equation suggest that, when the mean-free path and the associated relaxation time become large, the small disturbance propagation phenomena in gases is altered. At an altitude of 300 km, the particle number density is of the order of 10^9/cm^3, and the collision frequency is approximately 0.5 sec^{-1}. The mean-free path is consequently on the order of 1000 m. Nevertheless, on a surface of 1 cm^2, approximately 10^{14} collisions/sec occur. Hence a small body immersed in such a field can produce small disturbances that can propagate over very large distances. The superimposed effects of the resulting small force fields result in field phenomena that do not possess complete counterparts in classical hydrodynamics.

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Longitudinal and Transverse Propagation Phenomena

The propagation of small longitudinal disturbances in monatomic rarefied gas fields can be described by the equations recently developed by Grad.[7] Upon linearizing the three-dimensional, partial differential equations, characteristic solutions[2] can be obtained. For forward propagation, when $\tau_0 \gg L_0/(RT_0)^{1/2}$, the x components satisfy the following equations:

$$\frac{\partial}{\partial t}P_{1+}(x,t) + \left[\frac{5}{3}+\frac{1}{3}(10)^{1/2}\right]^{1/2}\frac{\partial}{\partial x}P_{1+}(x,t) = 0$$
$$\frac{\partial}{\partial t}P_{2+}(x,t) + \left[\frac{5}{3}-\frac{1}{3}(10)^{1/2}\right]^{1/2}\frac{\partial}{\partial x}P_{2+}(x,t) = 0 \quad (1)$$

where τ_0 is the relaxation time, and

$$P_{1+} = \theta_x + 0.63p_x + 1.04u + 0.66q_x + 1.03\sigma_x$$
$$P_{2+} = \theta_x - 0.63p_x - 0.49u + 0.31q_x - 0.23\sigma_x \quad (2)$$

Wu,[5] upon considering the problem of the flow generated by a suddenly heated plate and employing a two-sided Maxwellian, obtained a set of linearized equations which yield the following one-dimensional characteristic equations for forward propagating disturbances for $\tau_0 \gg L_0/(RT_0)^{1/2}$:

$$\frac{\partial}{\partial t}P_{1+}' + \left[\frac{5}{3}+\frac{1}{3}(10)^{1/2}\right]^{1/2}\frac{\partial}{\partial x}P_{1+}' = 0$$
$$\frac{\partial}{\partial t}P_{2+}' + \left[\frac{5}{3}-\frac{1}{3}(10)^{1/2}\right]^{1/2}\frac{\partial}{\partial x}P_{2+}' = 0 \quad (3)$$

where

$$P_{1+}' = 0.60N_1 + 0.30N_2 + 1.49t_1 + 0.15t_2$$
$$P_{2+}' = 0.58N_2 + 0.30t_1 + 0.07t_2 \quad (4)$$

N_1, N_2, t_1, and t_2 are the nondimensional perturbations of number density and temperature related by

$$\rho_x = \frac{N_1}{2}+\frac{N_2}{2} \qquad p_x = \frac{N_1}{2}+\frac{N_2}{2}+\frac{t_1}{2}+\frac{t_2}{2}$$
$$\theta_x = \frac{t_1}{2}+\frac{t_2}{2}$$
$$u = \frac{1}{(2\pi)^{1/2}}\left(N_1 - N_2 + \frac{t_1}{2} - \frac{t_2}{2}\right)$$
$$q_x = \frac{1}{(8\pi)^{1/2}}\left(\frac{7}{2}t_1 - \frac{7}{2}t_2 - N_1 + N_2\right) \quad (5)$$

and P_{1+}' and P_{2+}' can be written in the form

$$P_{1+}' = \theta_x + 0.63P_x + 1.04u + 0.66q_x$$
$$P_{2+}' = \theta_x - 0.63p_x - 0.49u + 0.31q_x \quad (6)$$

Hence the equations derived by Grad and Wu yield similar characteristic values, Eqs. (2) and (6), for small longitudinal disturbances that propagate along the same two distinct characteristics. The change in the propagation behavior from that occurring in the continuum limit is indicated by the fourth-order partial differential equation obtained by Wu for each of the dependent variables N_1, N_2, t_1, and t_2 represented by ϕ

$$\frac{\partial^4\phi}{\partial t^4} - \frac{10}{3}\frac{\partial^4\phi}{\partial t^2\partial x^2} + \frac{5}{3}\frac{\partial^4\phi}{\partial x^4} + \frac{\pi L_0}{6(RT_0)^{1/2}\tau_0}\left(\frac{\partial^3\phi}{\partial t^3} - \frac{5}{3}\frac{\partial^3\phi}{\partial t\partial x^2}\right) = 0 \tag{7}$$

When τ_0 is large, the expression multiplied by $1/\tau_0$ can be neglected, and the remaining terms describe the propagation of the small disturbances with the characteristic velocities $\pm [\frac{5}{3} \pm \frac{1}{3}(10)^{1/2}]^{1/2}$. When $\tau_0 \rightarrow 0$, in the continuum limit, the terms on the left may be neglected, and the small disturbances propagate with the velocity $\pm(\frac{5}{3})^{1/2}$, the dimensionless isentropic sound speed for a monatomic gas.

Independent "streams" can propagate in the x, y, and z directions in the large relaxation time limit. If a body submerged in the idealized field produces a steady thermal disturbance $P_+ = P_+(\theta)$,

$$\nabla \cdot \mathbf{P}_+ = 0 \qquad \mathbf{P}_+ = \mathbf{i}P_x(\theta) + \mathbf{j}P_y(\theta) + \mathbf{k}P_z(\theta) \tag{8}$$

Equation (8) suggests the existence of a continuous time-independent propagation of the thermal disturbance that is transmitted by the continuous motion of the field particles, i.e., a steady outgoing pressure disturbance occurs. For spherically symmetrical disturbances, $P_+ = \text{const}/r^2$, $r \geq a_1$. When a spherical body of radius a is submerged in the pressure field,[3] the resulting force exerted can be expressed in the form $F = \pm q^2/r^2$, $q^2 = \pi a_1^2 a^2 k p_0 \theta$. The force is positive or repulsive (negative or attractive) when $\theta > 0 (\theta < 0)$.

The equations developed by Lees[7] and Grad[8] suggest that small plane transverse disturbances can propagate in the large relaxation time limit. For forward propagating shear disturbances in the y direction, generated by small boundary motions along the x (or z) direction, the linearized first-order characteristic equation

$$\left[\frac{\partial}{\partial t} + \left(\frac{7}{5}\right)^{1/2}\frac{\partial}{\partial y}\right]Q_+ = 0 \qquad Q_+ = \sigma_{xy} + \left(\frac{7}{5}\right)^{1/2}\left(u + \frac{2}{5}q_x\right) \tag{9}$$

results from the Grad equations.[9] The shear equations suggest the existence of rarefied gasdynamic analogs of the electromagnetic equations describing the propagation of transverse disturbances in vacuum fields.[10]

Collision-Free Field Phenomena

Equations (1, 3, and 9), based on the analyses of Grad and Lees, suggest that similar equations can be obtained to describe small disturbance phenomena in the collision-free field. The equations can be developed by considering the Rayleigh problem.[11] Assuming that all field particles that collide with the surface are emitted with a Maxwellian distribution characteristic of the surface conditions, the distribution function solution of the one-dimensional Boltzmann equation is $f = f_0$ for $\eta \leq y/t$ and $f = f_0(1 + \epsilon_1 + \epsilon_2 + \epsilon_3)$ for $\eta > y/t$. When steady or slowly varying disturbances are produced by the boundary and the disturbances propagate over distances less than the mean-free path, the half-range solution becomes valid ($|y/t| \ll 1$, $\eta \rightarrow 0$). Mean values are then defined by

$$\bar{\phi} = \iiint_{-\infty}^{\infty} \phi f_0 d\xi d\eta d\zeta + \int_{-\infty}^{\infty}\int_0^{\infty}\int_{-\infty}^{\infty} \phi\epsilon_i f_0 d\xi d\eta d\zeta \tag{10}$$

Since the distribution function is linearized, the mean value equations satisfied by each of the disturbance energies ϵ_i can be obtained, and the conventional flow variables result from the superposition of the individual disturbance effects. When $\epsilon_i = \epsilon_1$, the two equations obtained for $\phi = 1$ and $\phi = \eta$ can be combined to yield the characteristic equation $(\partial/\partial t + c_1\partial/\partial y)\phi_1 = 0$, $\phi_1 = 3\rho_0\theta_w[1 + 8/(30\pi)^{1/2}]/4$ in the limit that the half-range solution is valid. Similarly, for the second longitudinal disturbance ϵ_2, the characteristic equation is $(\partial/\partial t + c\partial/\partial y)\phi_2$, $\phi_2 = -\rho_0\theta_w[1 + (2/\pi)^{1/2}]$. These equations suggest the existence of two longitudinal disturbances, which propagate with the adiabatic and isothermal velocities c_1 and c. Corresponding flow variables are $\rho = \rho_0 - \rho_0\theta_w/4$, $\rho v = 0$, and $P_{yy} = P_0 + P_0\theta_w/4$. First-order momentum propagation phenomena are not associated with thermal disturbance fields in this limit. When a small spherical body produces a thermal disturbance field, the resulting force field[12] is also of the form $F = \pm k'\theta_w/r^2$ as indicated in the previous section.

The corresponding characteristic equation ($\phi = \xi$ and $\xi\eta$) for the transverse disturbance ϵ_3 is $(\partial/\partial t + c\partial/\partial y)\phi_3 = 0$, $\phi_3 = \rho_0 u[1 + (2/\pi)^{1/2}]/2$. $\rho = \rho_0$, $\rho u = \rho_0 u/2$, and $P_{xy} = \rho_0 cu(2/\pi)^{1/2}/2$. The shear stress P_{xy} can produce a small force on a body of radius a, which is approximately equal to the drag experienced by the body when moving with velocity u,[13] i.e., the motion u can be transferred.

A spherical body rotating about a fixed axis with an angular velocity $\omega = v/a_1$ produces the shear field $P_{\phi r} = k''v/r$ in the plane of the equator.[14] If the rotating body also produces an attractive thermal force field, a small body of mass m can acquire the equilibrium motion $mv_0^2/r_0 = q^2/r_0^2$ in the plane of the equator when no other forces exist in the field. The drift velocity v_0 is not altered by the Brownian motion.[15] This centripetal relation can be expressed in the following forms: $r_0 = \beta^2/4\pi^2 mq^2$, $r_0\alpha = \beta/2\pi mc$, and $1/\alpha = \beta c/2\pi q^2$, where $\alpha = v_0/c$ and $\beta/2\pi = \alpha m r_0 c$. The orbital angular momentum may be written in the form $\beta/2\pi$ and the energy in the form $\beta\omega_0/2\pi$, where $\omega_0 = v_0/2r_0$.[16] The velocity acquired by m can be interpreted as resulting from n_0 field particle collisions per unit time which transmit the disturbance velocity v_0 at the radial distance r_0, $n_0 m_0 v_0 = mv_0$. For values of $r_i > r_0$, $v_i = v_0 r_0/r_i$ or $v_i = v_0 n_0'/n_0$ ($n_0' < n_0$). The difference in energy between any two orbits i and j is $E_{ji} = (2\pi^2 mq^4/\beta^2)(1/n_i^2 - 1/n_j^2)$, where n_i and n_j are the ratios of integers ($n_i < n_j$), and discrete orbits result.

The energy relations $mv_0^2/2 = q^2/2r_0$ and $mc^2/2 = q^2/2\alpha^2 r_0$ can be used as scale factors to relate the field energies to the total disturbance energy produced by the central body. The second-order energy $E_0' = mv_0^2/2$ is equal to $\alpha E_l'/2$ and to $\alpha E_t'/2$ where E_l' and E_t' are the mean first-order longitudinal and transverse energies for n_0 collisions at $r = r_0$. Further, if q^2 is assumed constant, when $r = 2a$ ($a = a_1$) the unbalanced longitudinal field energy $n_0 m_0 c^2/2 = mc^2/2$ is transmitted to m by n_0 collisions and $mc^2/2 = q^2/2a$. Hence, $\alpha^2 r_0 = a$ and the first-order field energies E_l and E_t can be written as $E_l = E_l'/\alpha^2$ and $E_t = E_t'/\alpha^2$. E_l corresponds to the energy transmitted across the area πa^2, which is $\frac{1}{4}$ of the energy emitted by the central body. The total transverse energy emitted is $\frac{2}{3}$ of this value since the velocity is not uniform over the surface. Hence, E_1, the total disturbance energy with respect to m emitted by the central body is $4(E_l + 2E_t/3) = \frac{20}{3}(2/\alpha)mc^2$. When small second-order corrections are neglected, the relations

$$E_0 = mc^2 = q^2/a \qquad \alpha^2 r_0 = a$$
$$E_1 = \tfrac{20}{3}(2/\alpha)mc^2 \qquad E_l = E_t = (2/\alpha)mc^2 \tag{11}$$

result. These classical relations are valid only when $\alpha = v_0/c \ll 1$ and may be of current interest since

$$E_1/c^2 = 1836.1m \qquad E_l/c^2 = 273.08m$$
$$E_t/c^2 = 264.08m \tag{12}$$

when small second-order corrections are included and $\alpha = \frac{1}{137.04}$.†

† To be submitted.

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Calculations of the Turbulent Boundary Layer

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VARIOUS authors (see, for example, Ref. 1) have proposed transformations by which a given compressible turbulent boundary-layer problem may be reduced to a corresponding problem in incompressible flow. This enhances the importance of having available accurate calculation methods for the incompressible boundary layer, and some results of a recent survey of existing methods by the writer may be of interest at this time.

A large number of calculation methods have been applied to cases where the boundary-layer development has been measured in (nominally) two-dimensional conditions. In most cases, the step-by-step solution of the momentum integral equation, using measured H values and a skin-friction law similar to that of Ludwieg and Tillmann,[2] disagreed noticeably with the measured development of momentum thickness as shown, for example, in Figs. 1 and 2 for the data of Refs. 8 and 9. This indicates the presence of three-dimensional flows in most of the measured layers, since the neglect of the turbulence terms in the momentum equation only becomes important close to separation.

Consequently, the various methods[3–7] that have been proposed for calculating momentum growth show no consistent relationship with experiment, since they cannot account for the range of crossflows (of either sign) that would be possible in principle for any given streamwise pressure distribution. Their relationship with the momentum integral solution suggests that a quadrature method with constants intermediate between those proposed by Spence[5] and Truckenbrodt[6] might be satisfactory in two-dimensional conditions, but the simplifying assumptions are always a source of uncertainty, and it would appear preferable to solve the integral equation directly.

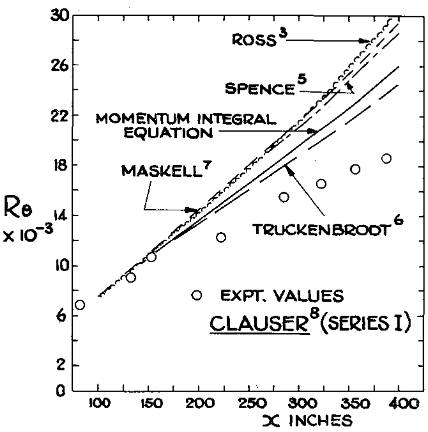

Fig. 1 Comparisons of calculated momentum thickness development.

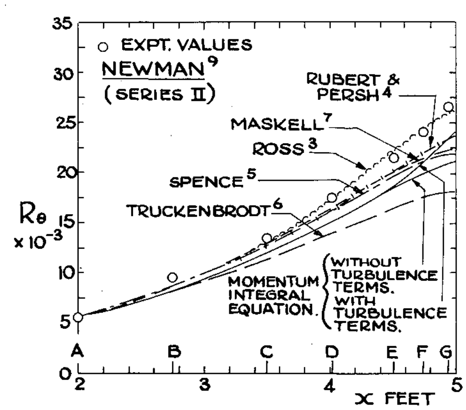

Fig. 2 Comparisons of calculated momentum thickness development.

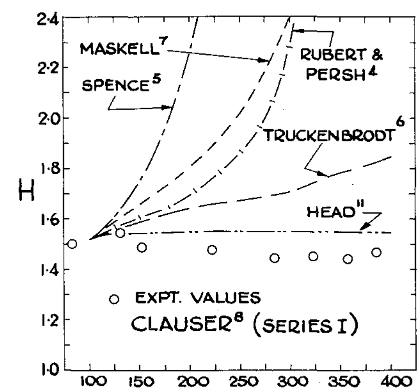

Fig. 3 Comparisons of calculated shape-factor developments.

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